PCA Quiz Questions

MCO

Question 1. We have a data set $\mathcal{D} = \{(1,2)^T, (7,3)^T, (3,4)^T, (-1,-1)^T\} \subset \mathbb{R}^2$. Let $\mathbf{X} = (X_1, X_2)^T$ be a r.v. with X_1 and X_2 random variables for the x_1 and x_2 co-ordinates respectively.

i) Find $\widehat{\mathbb{E}[\mathbf{X}]}$.

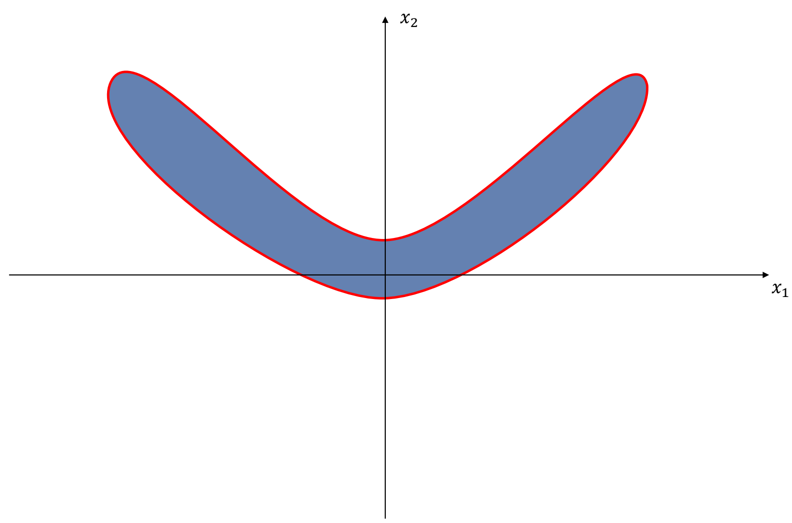
ii) Find $\widehat{Var[X_1]}$, $\widehat{Var[X_2]}$.

iii) Find $\widehat{Cov[X_1, X_2]}$.

(*give your answers in 3 significant figures*)

Question 2. We have a data set $\mathcal{D} = \{(-3,-7,-1)^T, (2,-1,-6)^T, (6,-2,6)^T, (-5,16,-1)^T, (-3,-6,4)^T, (2,-7,2)^T, (4,-1,-2)^T, (-3,8,-2)^T\} \subset \mathbb{R}^3$. Let $\mathbf{X} = (X_1, X_2, X_3)^T$ be a r.v. Find the co-variance matrix $Cov[\mathbf{X}]$ (*give your answers in 3 significant figures*).

Question 3. Suppose that we have a 2-dimensional data set $\mathcal{D}$ and we plot the points on a Cartesian graph, producing the following shape as shown:



Let $\mathbf{X} = (X_1, X_2)^T$ be a r.v. What do you expect the co-variance matrix $Cov[\mathbf{X}]$ to look like?

Question 4. Suppose we have the 3×3 square matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 4 & 5 \\ 3 & 2 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

Find all distinct eigenvectors of $\mathbf{A}$ and their respective eigenvalues (*give your answers in 3 significant figures*).

Question 5. Let $\mathcal{D}$ be a 2-dimensional data set of size N and $\mathbf{X} = (X_1, X_2)^T$ be a r.v, with X_1 and X_2 random variables for the x_1 and x_2 co-ordinates respectively. Suppose that for each X_i we have calculated the mean μ_i and variance σ_i using $\mathcal{D}$.

For $i = 1, 2$, Let $\widetilde{X}_i$ be a new r.v given by

$$\widetilde{X}_i = \frac{X_i - \mu_i}{\sigma_i}.$$

Show that $Cov[\widetilde{X}_1, \widetilde{X}_2] = Corr(X_1, X_2)$, where $Corr(X_1, X_2)$ is the sample correlation coefficient between X_1 and X_2 defined as

$$Corr(X_1, X_2) = \frac{Cov[X_1, X_2]}{\sigma_1 \sigma_2}.$$

Question 6. Suppose that we perform PCA on a d-dimensional data set $\mathcal{D}$, computing the eigenvectors $v_1, v_2, \ldots, v_d$ and their respective eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d$ in the process.

i) When projecting $\mathcal{D}$ onto a lower-dimensional data set, explain what it means when we want to preserve a proportion ρ of the variability in the data.

ii) Suppose that $d = 4$ and we find the following eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_4$ and their corresponding eigenvalues:

$$\lambda_1 = 2.97,$$
$$\lambda_2 = 1.17,$$
$$\lambda_3 = 0.64,$$
$$\lambda_4 = 0.2.$$

Which eigenvectors should be picked for our new axes so that 90% of the variation in $\mathcal{D}$ is preserved?

Question 7. In this question we will apply the PCA algorithm discussed in lectures using a 2-dimensional data set.

Let $\mathcal{D}$ be a data set given by

$$\begin{aligned}\mathcal{D} = \{&(-2,-3),(0,-2),(1,-1.5),(-1,-0.5),(1,-0.5),\\ &(0,-0.5),(4,-2),(1,0),(3,0),(3,1),(1,3),\\ &(3.5,3),(4,3),(5,1)\}.\end{aligned}$$

Let $\mathbf{X} = (X_1, X_2)^T$ be a r.v.

i) Find the co-variance matrix $\mathbf{C}$.

ii) Find the normalised eigenvectors of $\mathbf{C}$ and their corresponding eigenvalues. Write $\mathbf{C}$ in the form $\mathbf{V} \cdot \widetilde{\mathbf{C}} \cdot \mathbf{V}^T$, where $\mathbf{V}$ and $\widetilde{\mathbf{C}}$ are matrices as described in the lectures.

iii) Choose the eigenvector that will preserve the most variation in $\mathcal{D}$ and project the data points onto the new axis.

(*give your answers in 3 significant figures*)

Question 8. Let $\mathbf{C}$ be the co-variance matrix of a d-dimensional r.v $\mathbf{X} = (X_1, X_2, \ldots, X_d)^T$.

i) Show that $\mathbf{C} = \mathbb{E}(\mathbf{X}\mathbf{X}^T) - \boldsymbol{\mu}\boldsymbol{\mu}^T$, where

$$\boldsymbol{\mu} = \mathbb{E}(\mathbf{X}).$$

ii) Prove that the eigenvalues of $\mathbf{C}$ will always be positive.